Non-Abelian quantized Hall states of electrons at filling factors 12/5 and 13/5 in the first excited Landau level

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We present results of extensive numerical calculations on the ground state of electrons in the first excited ($n=1$) Landau level with Coulomb interactions, and including nonzero thickness effects, for filling factors 12/5 and 13/5 in the torus geometry. In a region that includes these experimentally relevant values, we find that the energy spectrum and the overlaps with the trial states support the previous hypothesis that the system is in the non-Abelian $k=3$ liquid phase we introduced in a previous paper.

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Many distinct quantum Hall liquid phases have been observed in two-dimensional electron systems. Leaving aside those that occur at integer filling factors (or quantized Hall conductance), the noninteger fractions occur at low filling factors in high mobility samples (at higher filling factors, they are supplanted by states in which translational or rotational symmetry is violated or by "re-entrant" integer quantized Hall phases). In the lowest ($n=0$) Landau level (LL), which is at filling factors of less than 2, the incompressible liquids are phases of matter that are correctly characterized by the Laughlin states,[1] and their extensions via the hierarchy[2] or composite fermion[3] approaches; these two approaches describe the same phases.[4] In the first excited ($n=1$) Landau level, the physics appears to be different, as one gets closer to the broken-symmetry phases. In general, fractions in the $n=1$ LL can be compared with those in the $n=0$ level by subtracting 2 from the filling factor; this corresponds to simply filling the lowest level with both spins and treating the next level like the lowest. A quantized Hall plateau occurs at fillings 5/2 and 7/2, which is now widely believed[5,6] to correspond to an incompressible liquid that may be viewed as a p-wave paired state of spin-polarized composite fermions in a half-filled LL.[7] It is unlike the lowest Landau-level (LLL) case, in which at filling factor $\nu=1/2$ or 3/2 the composite-fermion liquid appears so far to be gapless and exhibits Fermi-liquid-like properties.

A few other fractions are observed between 2 and 4, and one may wonder whether these are the same phases as occur in the LLL. In an earlier paper[8] (to be referred to as RR), a sequence of incompressible fractional quantum Hall liquids was constructed. These have filling factors $\nu=k/(Mk+2)$, where $k=1,2,3,\ldots$ and $M=0,1,2$ are integers, and for fermions M must be odd (the values $[(M-1)k+2]/(Mk+2)$ may also be obtained by using particle-hole symmetry). These were constructed within the LLL but may be applied to higher filling factors by adding the filling of the lower levels. The $k=1$ liquids are the familiar Laughlin states,[1] while $k=2$ is the Moore-Read (MR) paired state.[7] For the next case, $k=3$, some evidence that occurs in the $n=1$ Landau level (with $M=1$, so $\nu=2+\frac{3}{5}$ or $2+\frac{2}{5}$) was presented but was perhaps not entirely convincing. Meanwhile, experiments have observed a fractional quantum Hall state at 12/5, which exhibits a remarkably small energy gap for charged excitations.[9] There is a great deal of interest in phases such as the RR states, as for $k>1$ they exhibit excitations that obey non-Abelian statistics[7] and for $k\neq 1$, 2, or 4 they would support universal quantum computation.[10]

In this paper we return to the issue of the nature of the liquid state at filling fractions 12/5 and 13/5. We report numerical calculations for moderate numbers of electrons in a single LL, with interactions that model the Coulomb interaction between electrons in the $n=1$ LL, plus the effect of nonzero thickness of the electron wave function in the direction perpendicular to the two-dimensional layer. We also explore the phase diagram as the short-range component of the interaction is varied. For experimentally relevant values of thickness, we find that the liquid phase in the vicinity of the $n=1$ Coulomb interaction appears to be the RR $k=3$ phase. The evidence for this comes from the spectrum on the torus (i.e., periodic boundary conditions on a parallelogram), which exhibits a doublet of ground states that is characteristic of the $k=3$ phase, separated from higher excited states by a significant gap; this doublet does not occur for the hierarchy/composite-fermion (H-CF) phase. Quantitatively, the gap in the spectrum is small, suggesting that the gap for charged excitations will also be much smaller than for the H-CF 2/5 state in the LLL, in general agreement with experiment. The states in the ground-state doublet have large overlaps with the trial states of RR, adapted to the torus. When the interaction is modified, phase transitions to a "stripe" phase and to the usual H-CF phase are observed. Several of these results, except possibly for this last transition, are similar to results of a recent study[11] of bosons in the LLL at filling 3/2, which is the $k=3$, $M=0$ case of RR, and also to the 1/2 (5/2, etc.) case for electrons.[6]

The methods of calculation are rather standard, so we will be brief. For spin-polarized electrons in two dimensions confined to any one LL, the interaction Hamiltonian $H_{\rm int}$ in the infinite plane can be represented by the pseudopotentials V_m, $m=1,3,\ldots$. V_m is the interaction energy for a single pair of electrons of relative angular momentum m (m is odd).[2(a)] For the case of zero thickness, the pseudopotentials for the Coulomb interaction in the $n=1$ LL were plotted in Ref. 12. We also include nonzero thickness through the standard Fang-Howard method, in which the wave function contains the dependence $\propto x_3 e^{-x_3/(2b)}$ (but vanishes for $x_3<0$) on the per-

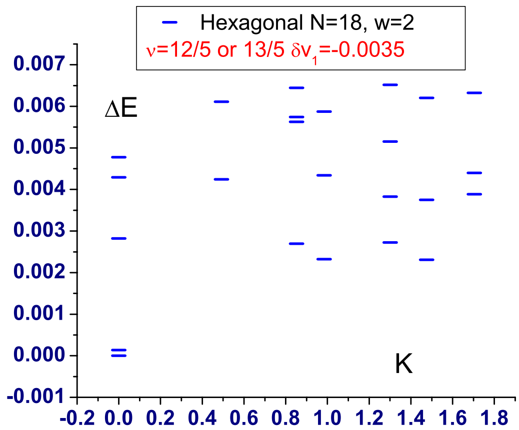

FIG. 1. (Color online) Low-lying spectrum for 18 electrons on the torus on the torus vs $K=|\mathbf{K}|$ for hexagonal unit cell. Energies are in units of e^2/ℓ_B. The interaction Hamiltonian is the $n=1$ Coulomb interaction, including nonzero thickness with parameter $w=2\ell_B$ and $\delta V_1=-0.0035$.

pendicular coordinate x_3. This tends to reduce somewhat the low m pseudopotentials relative to the others. We can then represent any one LL using the states in the LLL. This description of the interaction can be extended to the sphere (also using rotation symmetry) and to the torus; in these cases there are N_ϕ flux quanta piercing the system. Starting from the pseudopotentials for the nonzero thickness $n=1$ LL, we also consider changing the first two pseudopotentials V_1 and V_3 by small amounts δV_1 and δV_3 in order to explore the phase diagram in the vicinity of this interaction. This mitigates to some extent our ignorance of the precise interaction Hamiltonian. For comparison, for 15 particles on the torus and $w=2\ell_B$, the unperturbed values are $V_1=0.3858$ and $V_3=0.3333$. We also note that particle-hole symmetry holds as long as inter-LL interactions are neglected, as here, so that our results for $\nu=13/5$ also apply to 12/5, with no modifications at all in the case of the torus geometry. Likewise, our results should also be relevant for filling factors $3+\frac{2}{5}$ and $3+\frac{3}{5}$, which also lie in the $n=1$ LL. Accordingly, we refer only to $\nu=3/5$ from here on.

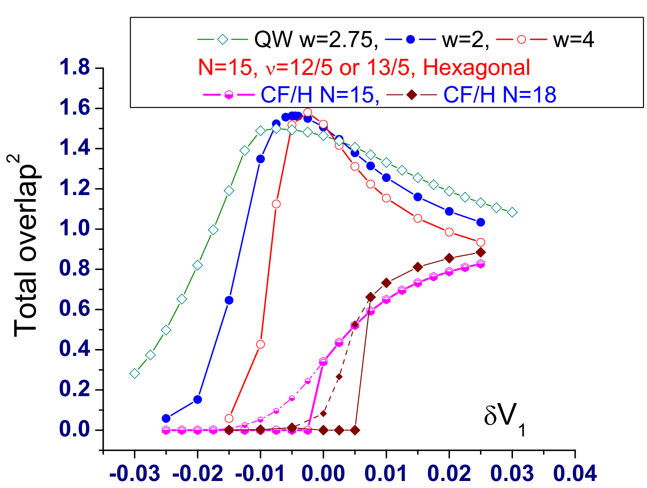

FIG. 2. (Color online) The sum of the squared overlap of each member of the ground-state doublet for $n=1$ Coulomb interaction (for $w=2$ and w) with the trial two-dimensional subspace, as a function of δV_1 on the torus, for 15 electrons. (Also included is the same for a quantum well wave function in x_3, labeled QW, with $w=2.75$.) Also, the squared overlap of one of the ground-state doublets for $w=2$ with the H-CF state, for both 15 and 18 electrons, as a function of δV_1 in the same system: solid line—lowest in doublet; dashed line—next lowest.

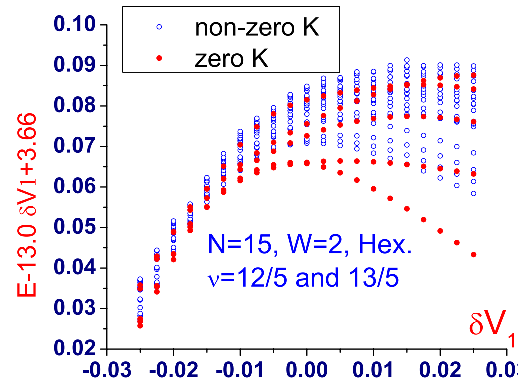

FIG. 3. (Color online) The low-lying energy spectrum as a function of δV_1 on the torus for 15 particles. The $\mathbf{K}=\mathbf{0}$ and $\mathbf{K}\neq\mathbf{0}$ levels are shown as distinct symbols.

We also refer to a (repulsive) p-body interaction which penalizes the closest approach of p fermions that is allowed by Fermi statistics; it can be written in terms of derivatives of δ functions on the sphere or torus. For fermions (electrons), the parafermion trial states found in Ref. 8 are unique, exact zero-energy eigenstates of such interactions for $p=k+1$ when N is divisible by k and $N_\phi=(k+2)N/k-3$ (on the sphere), so that $\nu=\lim_{N\to\infty} N/N_V=k/(k+2)$ (thus $M=1$). There are corresponding states on the torus for $N_\phi=(k+2)N/k$. For the $k+1$-body interaction, the trial states represent incompressible liquid phases, in which the excitations enjoy non-Abelian statistics for $k>1$. The $k+1$-body Hamiltonians allow us to numerically generate the trial states for comparison with the exact ground states of the two-body pseudopotential interaction at the same N, N_ϕ in the same geometry.

On the torus at $\nu=N/N_\phi=3/5$, translational symmetry implies that all energy eigenstates possess a trivial center-of-mass degeneracy of 5, which is exact for any size system, and also that there is a quantum number called $\mathbf{K}$,[13] which is a vector lying in a certain Brillouin zone. For $\mathbf{K}=\mathbf{0}$, energy eigenstates can also be labeled by their eigenvalues under a rotation. In the $M=1$ RR phases, the ground states have a net degeneracy $\frac{1}{2}(k+1)(k+2)$ in the thermodynamic limit, which is connected with the non-Abelian statistics.[7,8] For $k=3$, this tenfold degeneracy is made up of the trivial factor 5 (which is always discarded in numerical studies), together with a further twofold degeneracy, which in general becomes exact only in the thermodynamic limit; all these ground states have $\mathbf{K}=\mathbf{0}$. (For the four-body interaction, the tenfold degeneracy is exact for any size, as the ground states have exactly zero energy.) In contrast, the H-CF ground state for fermions at $\nu=3/5$ possesses only the fivefold degeneracy. Then for an incompressible fluid on the torus, the spectrum of a sufficiently large system in one of these two phases

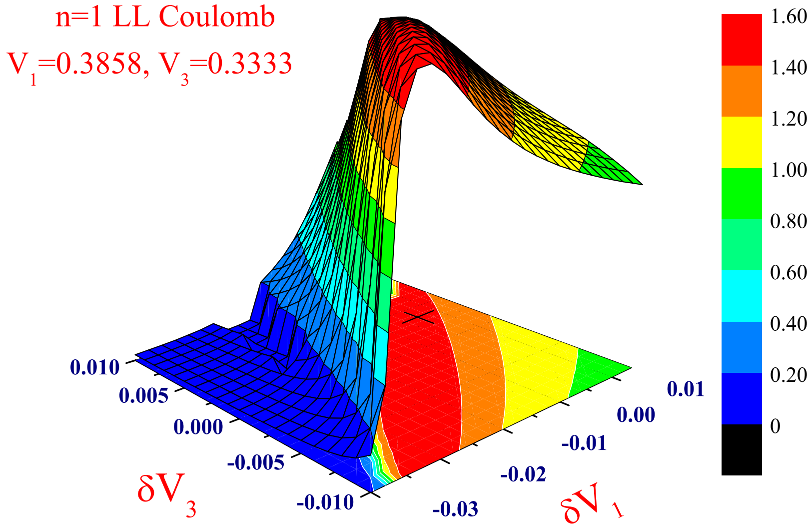

FIG. 4. (Color online) A three-dimensional plot of the total squared overlap with the RR trial subspace for $N=15$ particles on the torus, with $w=2$, as a function of both δV_1 and δV_3.

should exhibit a nearly degenerate pair of ground states or a single ground state at $\mathbf{K}=\mathbf{0}$, separated by a clear gap from a region of many states at higher energy eigenvalues.

In Fig. 1 we show the spectrum for 18 electrons on the torus at 30 flux quanta, with thickness parameter $w=2b=2$, and a small change to the pseudopotentials, $\delta V_1=-0.0035$ and $\delta V_3=0$. In this and all subsequent figures, the geometry of the torus is the hexagonal unit cell. Energies are in units of e^2/ℓ_B for the Coulomb interaction e^2/r. We set the magnetic length ℓ_B and $\hbar$ to 1 throughout. For this case, a ground-state doublet at $\mathbf{K}=\mathbf{0}$ is apparent, separated from the excited states by a gap which is about ten times larger than the splitting of the doublet or than the typical spacings above the gap. This is the expected behavior of the RR incompressible fluid phase and is similar to the case of bosons.[11] However, we will discuss below the variation in this splitting with the interaction parameters.

In Fig. 2, we show the sum of the squared overlaps of the lowest two $\mathbf{K}=\mathbf{0}$ states with the low-lying doublet in Fig. 1 with the two-dimensional subspace of zero-energy states of H_4 (the trial states) for $N=15$ particles, as a function of δV_1 ($\delta V_3=0$), for two values of $w=2b$. The same is also shown for another model of the nonzero thickness effect, which describes a quantum well in the x_3 direction, as used in some experimental samples; for this the thickness is $w=2.75$. In addition, we have plotted the squared overlap of each of the two states in the doublet with the single trial ground state for the H-CF phase for $w=2$. For the latter, we have used the numerically obtained ground state for a very large positive value of δV_1. In that regime, which is similar to the LLL Coulomb pseudopotential values, the ground state is believed to be incompressible and to lie in the H-CF phase. The H-CF state has exactly zero overlap with one member of the doublet, while the overlap of the other with the H-CF state is not zero. The overlap of the H-CF state with the lowest $\mathbf{K}=\mathbf{0}$ state drops abruptly to zero for δV_1 less than about 0, which is due to a level crossing. Beyond that point, the other $\mathbf{K}=\mathbf{0}$ state is lower and has zero overlap with the H-CF state, but the overlap of the other with the H-CF state remains nonzero and is shown as the dashed line. This behavior, both the vanishing overlap and the level crossing, indicates that the two members of the doublet have different rotational symmetry in the hexagonal geometry and only one has the same symmetry as the H-CF ground state. The nonzero overlap with the H-CF state is much smaller than that with the two-dimensional RR trial subspace but increases with increasing δV_1, while the overlaps with RR slowly decrease. (This behavior is consistent with that observed on the sphere at smaller sizes;[8] note that on the sphere the RR and H-CF states occur at different N_ϕ, which is not the case on the torus.) These results suggest that there must be a phase transition in the thermodynamic limit between these two incompressible phases at some value of δV_1, and this occurs for a range of values of thickness w.

In Fig. 3, we show the low-lying energy spectrum for $N=15$ particles as the δV_1 is varied away from zero for $w=2$ and $\delta V_3=0$. The level crossing of the lowest two $\mathbf{K}=\mathbf{0}$ levels, mentioned in the previous paragraph, can be seen at δV_1 close to 0. Over the approximate range $-0.01<\delta V_1<0$, the lowest two $\mathbf{K}=\mathbf{0}$ states stay remarkably close in energy and separated from all other states in the system. For negative δV_1, a transition occurs at around -0.01 (a similar transition was seen in Refs. 6 and 11). Beyond that point, the spectrum as a function of aspect ratio of the torus (not shown) shows signs that the system is in a stripe phase. Figure 3 reveals

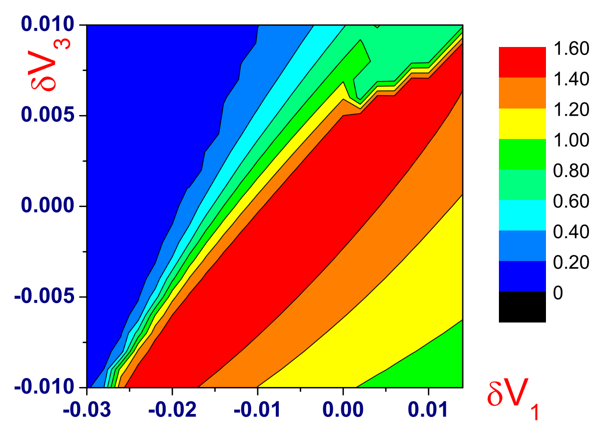

FIG. 5. (Color online) The same as Fig. 4 but shown as a contour plot.

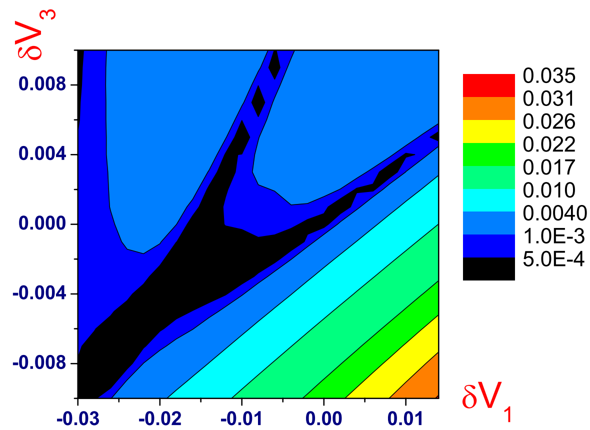

FIG. 6. (Color online) A contour plot of the level splitting between the two lowest $\mathbf{K}=\mathbf{0}$ states for 15 particles as a function of δV_1 and δV_3.

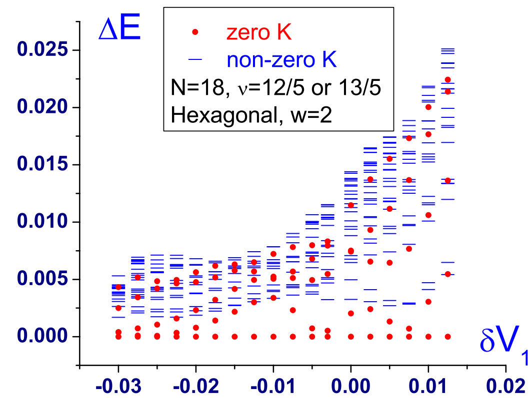

FIG. 7. (Color online) Low-lying energy spectrum, shown as the difference from the ground-state energy, for 18 particles as a function of δV_1. Again, states at $\mathbf{K}=\mathbf{0}$ and $\mathbf{K}\neq\mathbf{0}$ are shown with distinct symbols.

that this transition is the strongest feature in the spectrum, which dominates the behavior of moderately low-lying levels even rather far away in δV_1. The behavior of the spectrum, with many levels converging to zero at the same point, suggests that this transition may be second order in the thermodynamic limit. In addition, it indicates that the energy scales in the RR phase are smaller than in the H-CF phase which occurs at large positive δV_1. The transition between the latter two phases (where the higher $\mathbf{K}=\mathbf{0}$ state moves up into the continuum) occurs not far from the level crossing of ground states. In contrast to the rapid rise in energy of one of the two ground states of the RR region on entering the H-CF region, the higher energy levels show only gradual changes in this range of δV_1. This might suggest that this transition is first order in the thermodynamic limit, but see also the spectra for 18 particles below.

In Fig. 4 we show a three-dimensional plot of the total squared overlap with the RR trial subspace for $N=15$ particles on the torus, with $w=2$, as a function of both δV_1 and δV_3. This shows that the high, broad maximum as a function of δV_1 persists over a range of δV_3 though the location shifts. The same information is shown in Fig. 5 as a contour plot. We note that the squared overlap falls rapidly toward zero as the stripe region is entered. In view of the large dimensions of the Hilbert spaces of $\mathbf{K}=\mathbf{0}$ states, the largest squared overlaps (>1.40) should be viewed as significantly large.

In Fig. 6, we show the splitting between the two lowest $\mathbf{K}=\mathbf{0}$ states for 15 particles as a function of δV_1 and δV_3 also for $w=2$. For δV_3 zero or negative, the splitting is smallest close to where the overlap with the RR states is largest; however, at larger δV_3 the region of smallest splitting is seen to bifurcate, while the overlaps there are unaffected.

Finally, we show in Fig. 7 the low-lying spectrum for $N=18$ particles as a function of δV_1 for $\delta V_3=0$. Overall features are similar to those for 15 particles in Fig. 3; however, here the splitting of the lowest two $\mathbf{K}=\mathbf{0}$ states in the possible RR region is not as small as for $N=15$, and there are now two crossings between these two states between the strip region and the H-CF region (the spectrum in Fig. 1 is taken near one of these crossings). In addition, the gap to the nonzero $\mathbf{K}$ levels decreases somewhat in the transition region to the H-CF and is even comparable to the splitting of the doublet in that region (however, the gap to other $\mathbf{K}=\mathbf{0}$ states remains larger). Nonetheless, we expect that overlaps with the RR states remain large and those with the H-CF state remain small in the central region with a strong gap above the ground state. We note that the appearance of two level crossings, with a similar not-so-small splitting between them, also seems to occur in the $N=15$ case if we examine larger values of δV_3 (the "bifurcation" noted above), while overlaps with the RR subspace remain large. Consequently, this aspect of the $N=18$ data seems consistent with behavior at $N=15$.

Uniting all aspects of our results, we conclude that over a range of parameter values that includes the $n=1$ Coulomb interaction and experimentally relevant nonzero thickness effects, there is a large overlap of the lowest two $\mathbf{K}=\mathbf{0}$ states with the trial subspace and relatively small splittings of those two states, suggesting that they become degenerate in the thermodynamic limit. This region is bordered on one side by the H-CF phase, which has poor overlap (and for some parameters *zero* overlap due to different symmetry) with the lowest doublet in this region. It is bordered on the other side by a stripe phase, similar to others in the higher LLs. The overall pattern of behavior is quite similar to that observed for the MR phase at 5/2 (Ref. 6) and is also consistent with our previous results on the present system on the sphere.[8]

To conclude, we find the evidence that the observed 12/5 state is in the RR phase compelling. The nonobservation of the 13/5 state so far suggests that LL mixing plays a role at the latter filling factor. While a systematic study of energy gaps for charged excitations will have to await larger system sizes, we find indications that the scales in this state will be smaller than in the H-CF phase at the same density of particles, which is broadly consistent with experimental findings.[9]

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